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"Different Method's for Solving Fractional Differential Equations and their Applications"

Research proposal submitted

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Abstract:

We discuss standard approaches to the problem of fractional derivatives and fractional integrals (simply called differintegrals), namely the Riemann-Liouville, the Caputo and the sequential approaches. We prove the basic properties of differintegrals including the rules for their compositions and the conditions for the equivalence of various definitions. Further, we give a brief survey of the basic methods for the solving of linear fractional differential equations and mention limits of their usability. In particular, we formulate the theorem describing the structure of the initial-value problem for linear two-term equations. Finally, we consider some physical applications, in particular fractional advection- dispersion equation and the viscoelasticity problem. Dealing with the first issue we derive the fundamental solution in the form of Levy stable distribution density and then we discuss relations between the generalized central limit theorem and the choice of the corresponding fractional model. In section concerning viscoelasticity we mention some typical models generalizing the standard ones via the replacement of the classical terms by the fractional terms. In particular, we derive the step response functions of those generalized systems.

I. Introduction to proposed research topic:

In recent years, fractional order calculus has been studied as alternative calculus in mathematics and it was applied in various field in science and engineering. Numerous problems in Physics, Chemistry, Biology and Engineering can be models with fractional derivatives. Now the most of modeled fractional differential equations donot have analytic solutions. Therefore, numerical techniques are used intensively, recently the adomian decomposition method, homotopy perturbation method and variational iterative methods have been used for solving wide range of problems. In our work we shall develop new iterative methods such as fractional order Adomian decomposition method, fractional order variational iteration method and explicit fractional order finite difference methods to obtain the numerical solutions of fractional differential equations. Many researcher using definition of fractional derivative and some different methods for solving to Fractional Differential Equation:

Definition of the Riemann Liouville Fractional Integral:

The Riemann–Liouville definition. For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

Definition of the Caputo's Fractional Derivative :

Caputo definition. For α ∈ [n−1, n), the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\int_a^{\alpha} \frac{f^{(n)}(x)}{(\mathrm{t}-\mathrm{x})^{\alpha-\mathrm{n}+1}}dx$$

1) Homotopy Analysis Method

A homotopy between two continuous function $f(x)$ and $g(x)$ from a topological space X to a topological space Y is formally defined to be a continuous function $\mathcal{H}: X * [0,1] \to Y$ from the product of the space X with the unit interval [0,1] to Y such that , it x in X then $H(x;0) = f(x)$ and $\mathcal{H}(x;1) = g(x)$.

2) Novel Numerical Method:

Suppose the two dimensional Laplacian has a complete set of orthonormal eigen functions $\varphi_{n,m}$ corresponding to eigen values $\lambda_{n,m}$ in a rectangular region

$D = \{(x,y)|0 \le x \le a, 0 \le y \le b\}$, $B(\varphi) = 0$ on ∂D, where $B(\varphi)$ is the homogeneous dirichlet boundary condition.

3) Fractional differential transform method:

There are several approaches to the generalization of the notion of differentiation to fractional orders. The fractional differentiation in Riemann–Liouville sense is defined by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}dx$$

For $m-1 \le q < m, m \in Z^+, x > x_0$ let us expand the analytical and continuous function f(x) in terms of a fractional power series as follows

$$f(x) = \sum_{k=0}^{\infty} F(x)(x-x_0)^{k/\alpha}$$

Where α is the order of fraction and F(k) is the fractional differential transform of f(x). concerning the practical applications encounterd in various brances of science, the fractional initial conditions are frequently not available, and it may not be clear what their physical meaning is. Therefore, the definition in above equation should be modified to deal with integer ordered initial conditions in Coputo sense as follows:

$$D_{x_0}^{q}\left[f(x)-\sum_{k=0}^{\infty}\frac{1}{k!}(x-x_0)^{\frac{k}{\alpha}}f^{k}(0)\right]=\frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\left\{\int_0^x\left[\sum_{k=0}^{\infty}\frac{1}{k!}(x-x_0)^{\frac{k}{\alpha}}f^{k}(0)\right]dt\right\}$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$\mathrm{F(k)}=\begin{cases}\mathrm{If}\frac{\mathrm{k}}{\alpha}\in Z^{+}\ then\ \frac{1}{(k/\alpha)}\left[\frac{d^{\frac{k}{\alpha}}f(x)}{dt^{\frac{k}{\alpha}}}\right]_{x=x_0},\ \text{for } k=0,1,2,\dots,(n\alpha-1)\\ \mathrm{If}\frac{\mathrm{k}}{\alpha}\ not\ in\ Z^{+}\ then\ \ 0\end{cases}$$

where, n is the order of FDE considered

For example, we will solve the following types of fractional partial differential equations

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\alpha\frac{\partial^{2}u(x,t)}{\partial x^{2}}+f(x,t),\qquad 0\le x\le L,0<\alpha\le 1,t>0$$

Initial condition: $u(x,0)=f(x),0\le x\le L$

Boundary condition: $u(0,t)=g_{(1),}\,u(L,t)=g_2(t),t>0$

Purpose of carrying out proposed research work:

Differential equations play a major role in Mathematics, Physics and Engineering. The purpose of the research will develop different methods for solving fractional differential equations to obtain their solutions. Also, we will represent graphically. The solution of fractional differential equations obtains by new iterative methods using mathematical software Mathematica.

Also we will study Numerical methods to obtain the solutions of fractional partial differential equations, such as Explicit methods. We will compare the solutions obtain by Decomposition methods and Numerical methods.

II. Review of literature and development in the relevant subject

A numerical method based on the Adomian Decomposition Method (ADM) which has been used from the 1970's to the 1990's by George Adomain, The Differential Transform Method (DTM) has been successfully developed by Zhou (1986) in electric circuit analysis. DTM has been used to solve linear and non-linear differential equations.

ADM and DTM has been shown to solve effectively, easily and accurately a large class of linear and non-linear, ordinary, partial deterministic or stochastic differential equations with approximate solutions which converge rapidly to accurate solutions. The basic motivation of the work is to apply the ADM and DTM to the ordinary differential equations (ODE) and partial differential equations (PDE). It is well known now in the literature that this algorithm provides the solution in a rapidly convergent series. ADM and DTM are very effective and

convenient for solving multi-pantograph equations.

Variational Iteration Method (VIM) is first proposed by Ji-Huan He. In the year 1998, to solve fractional differential equations with great success following the above idea, Draganescu, Momani and Odibate applied VIM to more complex fractional differential equations, showing the effectiveness and accuracy of the method.

In 2002, the Adomain Decomposition Method (ADM) was suggested to solve fractional differential equations but many researches found it very difficult to calculate the Adomain polynomials. In 2007, Momani and Odibate applied the Homotopi - Perturbation Method (HPM) to fractional differential equations and show that HPM is an alternative methods for obtaining the solutions of fractional differential equations.The another powerful method is called the Variational Iteration Method (VIM) has been successfully applied to many situations.

Recently, a new iterative method was presented by Daftardar – Gejji and Jafari. This technique solves many types of non linear equations such as ordinary and partial differential equations of integer and fractional order. Jafari et al applied. This method to obtain the solution of linear / non-linear diffusion and wave fractional equations. Daftardar – Gejji and Bhalekar used it for solving fractional boundary value problems and evolutionequations.

Numerical methods and analysis of the fractional order partial differential equations are limited to date. Some different numerical methods for solving the space or time fractional partial differential equations have been proposed Liu et al transformed the space fractional partial differential equation in to a system of ordinary differential equations (Method of Lines) which was then solved using backward differentiation formulae.

Shen et al proposed an explicit finite difference approximation for the space fractional diffusion equation the stability and convergence of the method is discussed Lin and Liu proposed the high order approximation of the fractional ordinary differential equation and discussed the convergence and stability of the fractional high order methods.

S. P. Goyal and T. Mathur (2003), Recently several researchers have established interesting relationships between the Laplace transform, the Varma transform and the general Weyl fractional integral. Motivated by their work, in this paper authors establish a theorem which asserts an interesting relationship between the multidimensional Laplace transform, the multidimensional Varma transform and the generalized Weyl fractional integral involving product of a general class of multivariable polynomials and a generalized polynomial set. By specializing the various parameters involved, this general theorem would readily yield several (known and new) results involving simpler integral operators. Further, the theorem is applied

to evaluate the generalized Weyl fractional integrals of Fox's H-function and the (Srivastava – Panda) H-function of several complex variables.

R. K. Saxena, R. K. Yadav, S. D. Purohit and S. L. Kalla (2005) In this paper authors evaluated Kober fractional basic integral operator of the basic analogue of the H-function. The results obtained are believed to be a new contribution to the theory of q-fractional calculus and are likely to find certain applications to the solution of the fractional q-integral equations involving various q-hypergeometric functions.

V. B. L. Chaurasia and A. Srivastava (2007), A lot of research work has recently come up on the study and development of fractional calculus operators to obtain image formulas in terms of several special functions like Meijer's G- Function, Fox's H- Function and the H □-Function that is a generalization of Fox's H-Function. Motivated by these avenues of applications, in this paper authors study a pair of unified and extended fractional integral operators involving the multivariable H-Function, Fox's H-Function and general class of polynomials. They also establish some theorems pertaining to Mellin transforms of these operators and, some properties of these operators have been investigated.

H. J. Haubold, A. M. Mathai and R. K. Saxena (2009), Motivated essentially by the success of the applications of the Mittag-Leffler functions in many areas of science and engineering, the authors present in a unified manner, a detailed account or rather a brief survey of the Mittag-Leffler function, generalized Mittag-Leffler functions, Mittag-Leffler type functions, and their interesting and useful properties. Applications of Mittag-Leffler functions in certain areas of physical and applied sciences are also demonstrated. In this survey paper, nearly all types of Mittag-Leffler type functions existing in the literature are presented. An attempt is made to present nearly an exhaustive list of references concerning the Mittag-Leffler functions to make the reader familiar with the present trend of research in Mittag-Leffler type functions and their applications.

S. D. Purohit and R. K. Yadav (2010), The fractional q-calculus is the q-extension of the ordinary fractional calculus. The theory of q-calculus operators in recent past have been applied in the areas like ordinary fractional calculus, optimal control problems, solutions of the q-difference (differential) and q-integral equations, q-transform analysis etc. Recently several workers deduced several transformations and summations formulae for the basic hypergeometric functions as the applications of fractional q-differential operator. In the paper, authors introduced two hypergeometric operators of fractional q-integration, which may be regarded as extensions of the fractional q- integral operators. By defining q-extensions of these operators, authors investigated their fundamental properties such as integration by parts and

connection theorem with q-analogue of Mellin Transform.

V. B. L. Chaurasia and Y. Singh (2011), Authors established an integral concerning the product of generalized Lauricella function and two H-function of several complex variables using the series representation of H-function. Further, by giving suitable values to the parameters, the main integral reduces to F function.

M. Chand (2012), In the paper, the author establish three new theorems, which involves I-function and general class of polynomials. Next, he obtain certain new integrals and expansion formulas by the application of these theorems. By giving suitable values to the parameters, the main integral reduces to Fox's H-function, G-function and generalized wright hypergeometric function, etc. which provide certain results (known and new).

V. B. L. Chaurasia and V. Gill (2013), In the paper an equation of Internal Blood Pressure pertaining to the Aleph-function is established. As the Aleph-function is the most generalized special function, numerous special cases with potentially useful transcendental functions, several useful results involving certain special functions can be obtained by the main result.

III. Research gap:

Most important thing is that many fractional differential equations do not have an analytical solution. Therefore, we develop the numerical methods to obtain there solutions. Also, to discuss the stability and convergence of the numerical methods.

Also wide scope in to obtain exact solution of fractional differential equations by developed new different methods and it will represent graphically.

IV. Importance of the Study:

Fractional calculus is becoming a useful and key tool in the analysis of scientific problems in Physics, Engineering, Biology and Economics, Fractional formalism to be useful in solving practical problems, fractional calculus, integration and differentiation of an arbitrary or fractional order, provides new tools that expand the descriptive power of calculus beyond the familiar integer-order concept of rate of change and area under thecurve.

a) International Status: Many scientists and researchers many more contributed in the development of fractional differential equations and Fractional calculus like, Leibniz, Euler, Lagrance, Laplace, Fourier, Abel, Liouvielli, Grunwald, Riemann, Riesz, Miller, Ross, Igor Prodlubny, Richtmeyer, Morton, Smith, Anderson, Roache etc.

b) National Status: Indian's also contributed remarkable in the development of fractional calculus like R.K. Sexana, V.Srivastava, K.N. Rai, S. Das, K. Gupta, S. Saha Roy, V.B.L. Chawrasia, D. Kumar, Pradeep Kumar, D. Bahuguna etc.

c) Regional Status: Many researchers from Maharashtra have been actively involve in the study of fractional calculus. Like Kiran, M. Kolwankar, A.K. Pani, PSV Nataraj, S. Bhalekar, V. Daftardar- Gejji, K.C. Takale, J. Verma, D.B. Dhaigudeetc.

d) Scope of Study: The analysis of different methods for solving fractional differential equations and their applications. During the last decade, fractional calculus has been applied to almost every field of Science, Engineering and Mathematics. Many application of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems and astrophysics. There are few different methods for obtaining the solution of fractional differential equation (FDE), therefore, we develop new different methods to obtain their solutions and discuss the stability and convergence of the numerical methods upto the length. We will study the powerful mathematical software Mathematica to simulate the solutions graphically. Further we will study fractional wave equations, fractional differential equations and fractional partial differential equations. We will develop iterative and finite difference schemes to solve them. Solutions of these equations will be simulated graphically. We intend to publish the work in reputed Journals.

V. Objectives of Proposed Research Work:

1) To study linear and nonlinear ordinary and partial differential equations.

2) To study different methods for solving fractional differential equations and their applications.

3) To study basic concept of fractional calculus.

4) To study analytical and numerical methods for solving linear and nonlinear fractional partial differential equations.

5) To develop different methods for solving fractional differential equations.

6) To obtain analytical solutions of fractional differential equations. we obtain the approximate solutions of fractional partial differential equations by developed fractional order finite difference schemes and compare these solutions with their analytical solution, if itis possible.

7) To discuss convergence and stability of solutions obtained by developed numerical methods.

8) To stimulate the solutions of fractional partial differential equations using mathematical software Mathematica and to discuss the nature of the solutions.

9) To study the recent applications of different methods for solving fractional differential

equations and fractional partial differential equations in a various fields of science and technology. To publish research papers in reputed journals.

VI. Hypothesis:

The proposed work is to prove that how mathematics is useful in other branches of science. We hope that our research work will be useful in applied sciences and engineering field.

Hence, in the present work, we will try to develop the new iterative techniques for fractional differential equation and to obtain their solution. These will represented graphically by Mathematica software.

VII. Limitations of the work:

1) The fractional order model are more adequate than integer order model.

2) Fractional derivatives and integrals enable the description of memory properties of various materials and processes.

3) Most of the fractional differential equations do not have analytical solutions.

VIII. Research Methodology:

The following steps will be including in the methodology.

1) To refer the relevant reference books on partial differential equation and fractional partial differential equation.

2) To study Mathematica software.

3) To study fractional calculus.

4) To study Iterative / Numerical methods for obtaining approximate solutions of fractional differential equations.

5) To discuss the stability and convergence of Iterative / Numerical methods.

6) To represent Numerical solutions of fractional differential equations graphically by Mathematica and discuss their nature.

IX. Expected outcome:

1) We will develop different methods for solving fractional differential equations and their comparison.

2) We will develop numerical methods for fractional partial differential equations.

3) To find the approximate solutions of fractional differential equations by developed new iterative technique and it will represented graphically by Mathematica software.

X. Year wise plan of work and targets to be achieved:

First year plan:

1) To refer the reference books on differential equation and fractional differential equation.
2) To develop different methods, Numerical methods for differential equation.
3) To obtain the solutions of partial differential equations by iterative methods.
4) To visualize solutions by graphs using a powerful software Mathematica and MATLAB.

Second year plan:

1) To study fractional calculus.
2) To study fractional differential equations.
3) To develop different methods for obtain the solution of fractional differential equations.
4) To visualize the solutions by graphs using Mathematica, MATLAB and discuss the nature.

Third year plan:

1) To develop different methods for fractional partial differential equations.
2) To discuss stability and convergence of the numerical methods.
3) We will publish our work in reputed journals.
4) Submission of completed work to concerned authority.

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